Senior Mathematical Challenge

Organised by the United Kingdom Mathematics Trust

supported by Institute and Faculty of Actuaries

Solutions and investigations

7 November 2017

These solutions augment the printed solutions that we send to schools. For convenience, the solutions sent to schools are confined to two sides of A4 paper and therefore in many cases are rather short. The solutions given here have been extended. In some cases we give alternative solutions, and we have included some problems for further investigation. We welcome comments on these solutions. Please send them to enquiry@ukmt.org.uk.

The Senior Mathematical Challenge (SMC) is a multiple-choice paper. For each question, you are presented with five options, of which just one is correct. It follows that often you can find the correct answers by working backwards from the given alternatives, or by showing that four of them are not correct. This can be a sensible thing to do in the context of the SMC.

However, this does not provide a full mathematical explanation that would be acceptable if you were just given the questions without any alternative answers. So we aim at including for each question a complete solution with each step explained (or, occasionally, left as an exercise), and not based on the assumption that one of the given alternatives is correct. We hope that these solutions provide a model for the type of written solution that is expected when presenting a complete solution to a mathematical problem (for example, in the British Mathematical Olympiad, the Mathematical Olympiad for Girls and similar competitions).

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1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25
C	B	E	E	C	B	C	B	B	D	D	B	C	A	B	D	D	E	B	B	C	D	C	D	A

1. One of the following numbers is prime. Which is it?

A $2017-2$ B $2017-1$ C 2017 D $2017+1$ E $2017+2$

SOLUTION **C**

We see that:

$2017-2=2015$ and 2015 is a multiple of 5. So $2017-2$ is not prime.

$2017-1=2016$ and 2016 is a multiple of 2. So $2017-1$ is not prime.

$2017+1=2018$ and 2018 is a multiple of 2. So $2017+1$ is not prime.

$2017+2=2019$ and 2019 is a multiple of 3. So $2017+2$ is not prime.

We are told that one of the given options is prime. We may therefore deduce that the remaining option, 2017, is prime.

FOR INVESTIGATION

1.1 In the context of the SMC it is safe to assume that the information given in the question is correct. Therefore, having shown that $2017-2$, $2017-1$, $2017+1$ and $2017+2$ are not prime, we may deduce that 2017 is prime.

However, without this assumption, if we wish to show that 2017 is prime, we need to check that there are no prime factors of 2017 which are less than 2017.

Which is the largest prime p that we need to check is not a factor of 2017 in order to show that 2017 is prime?

1.2 One way to see that 2019 is a multiple of 3 is by noting that the sum of its digits, $2+0+1+9=12$, is a multiple of 3.

Why does this test for divisibility by 3 work?

1.3 (a) Which is the least positive integer n such that either $2017-n$ or $2017+n$ is prime?

(b) Which is the least positive integer n such that both $2017-n$ and $2017+n$ are prime?

2. Last year, an earthworm from Wigan named Dave wriggled into the record books as the largest found in the UK. Dave was 40 cm long and had a mass of 26 g.

What was Dave's mass per unit length?

A 0.6 g/cm B 0.65 g/cm C 0.75 g/cm D 1.6 g/cm E 1.75 g/cm

SOLUTION **B**

Dave's mass per unit length is $\frac{26}{40}$ g/cm. We have

$$\frac{26}{40}=\frac{26}{10\times 4}=\frac{2.6}{4}=0.65.$$

Therefore Dave's mass per unit length is 0.65 g/cm.

3. The five integers 2, 5, 6, 9, 14 are arranged into a different order. In the new arrangement, the sum of the first three integers is equal to the sum of the last three integers.

What is the middle number in the new arrangement?

A 2 B 5 C 6 D 9 E 14

SOLUTION **E**

Let the integers in the new arrangement be in the order p, q, r, s, t. Because the sum of the first three integers is the same as the sum of the last three,

$$p + q + r = r + s + t,$$

and hence

$$p + q = s + t.$$

We therefore see that the pair of integers p, q has the same sum as the pair s, t. Also, the middle number, r, is the one that is not included in either of these pairs.

It is straightforward to check that $2 + 9 = 5 + 6$ and that 2, 9 and 5, 6 are the only two pairs of the given integers with the same sum.

Therefore the middle integer in the new arrangement is 14, as this does not occur in either pair.

FOR INVESTIGATION

3.1 In how many different ways may the integers 2, 5, 6, 9, 14 be arranged into a different order so that the sum of the first three integers is equal to the sum of the last three integers?

3.2 Suppose that the integers 3, 7, 8, 10, 12 are arranged into a different order so that the sum of the first three integers is equal to the sum of the last three. What is the middle number in the new arrangement?

3.3 The integers 3, 6, 9, 12, 15 are to be arranged into a different order so that the sum of the first three integers is equal to the sum of the last three. How many different possibilities are there for the middle number in the new arrangement?

3.4 Five different integers are to be arranged in order so that the sum of the first three integers is the same as the sum of the last three. What is the maximum number of possibilities for the middle number in the new arrangement?

3.5 (a) What is the largest number of integers that may be chosen from the set of all positive integers from 1 to 10, inclusive, so that no two pairs of the chosen integers have the same total?

(b) What is the largest number of integers that may be chosen from the set of all positive integers from 1 to 20, inclusive, so that no two pairs of the chosen integers have the same total?

4. Which of the following is equal to $2017 - \dfrac{1}{2017}$?

A $\dfrac{2017^2}{2016}$ B $\dfrac{2016}{2017}$ C $\dfrac{2018}{2017}$ D $\dfrac{4059}{2017}$

E $\dfrac{2018 \times 2016}{2017}$

Solution **E**

Writing both 2017 and $\dfrac{1}{2017}$ over a common denominator, we have

$$2017 - \frac{1}{2017} = \frac{2017^2 - 1}{2017}.$$

Now,

$$2017^2 - 1 = 2017^2 - 1^2.$$

Hence, using the standard factorization of the difference of two squares, we have

$$2017 - \frac{1}{2017} = \frac{2017^2 - 1^2}{2017} = \frac{(2017 + 1)(2017 - 1)}{2017} = \frac{2018 \times 2016}{2017}.$$

5. One light-year is nearly 6×10^{12} miles. In 2016, the Hubble Space Telescope set a new cosmic record, observing a galaxy 13.4 thousand million light-years away.

Roughly how many miles is that?

A 8×10^{20} B 8×10^{21} C 8×10^{22} D 8×10^{23} E 8×10^{24}

Solution **C**

One thousand million is $1000 \times 1\,000\,000 = 10^3 \times 10^6 = 10^{3+6} = 10^9$. Therefore 13.4 thousand million light-years is 13.4×10^9 light-years. Therefore, because a light-year is nearly 6×10^{12} miles, 13.4 thousand million light-years is approximately

$$(6 \times 10^{12}) \times (13.4 \times 10^9) \text{ light-years.}$$

Now

$$(6 \times 10^{12}) \times (13.4 \times 10^9) = (6 \times 13.4) \times (10^{12} \times 10^9).$$

Now 6×13.4 is approximately 80, therefore, $(6 \times 13.4) \times (10^{12} \times 10^9)$ is approximately

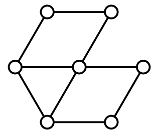
$$80 \times (10^{12} \times 10^9).$$

Finally, we have

$$80 \times (10^{12} \times 10^9) = 8 \times 10 \times 10^{12+9} = 8 \times 10 \times 10^{21} = 8 \times 10^{22}.$$

Therefore 13.4 thousand million light-years is approximately 8×10^{22} miles.

6. The circles in the diagram are to be coloured so that any two circles connected by a line segment have different colours.

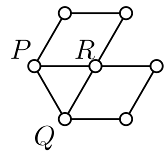


What is the smallest number of colours required?

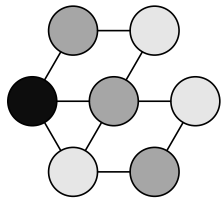
A 2 B 3 C 4 D 5 E 6

Solution **B**

Each pair of the circles labelled P, Q and R in the figure on the right is connected by a line segment. Therefore these three circles must be coloured using different colours. So at least three colours are needed.



The figure on the right (with the circles enlarged for the sake of clarity) shows one way to colour the circles using three colours so that circles connected by a line segment have different colours.

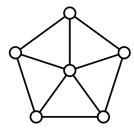


Therefore 3 is the smallest number of colours required.

For investigation

6.1 In how many different ways is it possible to colour the circles in the diagram in the question, using the three colours red, green and blue, so that circles connected by a line segment have different colours?

6.2 What is the smallest number of colours needed to colour the circles in the figure on the right so that circles connected by a line segment have different colours?



6.3 It follows from the *Four Colour Theorem* that any arrangement of circles in the plane, connected by line segments that do not cross one another, may be coloured using at most four colours so that circles connected by a line segment have different colours.

Find an arrangement of circles connected by line segments for which four colours are needed.

What is the smallest number of circles in such an arrangement?

Note

The first proof of the Four Colour Theorem about maps drawn in the plane was published by Kenneth Appel and Wolfgang Haaken in 1977. Their proof reduced the general case to 1482 *unavoidable* configurations which needed to be checked separately. These configurations were generated and checked by a computer program. Since 1977 simpler proofs using a computer have been found. But no-one has yet found a proof which is simple enough for a human being to check it, just using pencil and paper, in a reasonable amount of time.

A good book on the Four Colour theorem is *Four Colours Suffice: How the Map Problem was Solved*, Robin Wilson, 2002.

7. The positive integer k satisfies the equation $\sqrt{2} + \sqrt{8} + \sqrt{18} = \sqrt{k}$.

What is the value of k?

A 28 B 36 C 72 D 128 E 288

Solution **C**

Because $8 = 2^2 \times 2$ and $18 = 3^2 \times 2$, we have $\sqrt{8} = 2\sqrt{2}$ and $\sqrt{18} = 3\sqrt{2}$. Therefore

$$\begin{aligned}\sqrt{2} + \sqrt{8} + \sqrt{18} &= \sqrt{2} + 2\sqrt{2} + 3\sqrt{2}\\ &= 6\sqrt{2}\\ &= \sqrt{6^2 \times 2}\\ &= \sqrt{72}.\end{aligned}$$

Therefore $k = 72$.

8. When evaluated, which of the following is not an integer?

A 1^{-1} B $4^{-\frac{1}{2}}$ C 6^0 D $8^{\frac{2}{3}}$ E $16^{\frac{3}{4}}$

Solution **B**

We have

$$\begin{aligned}&1^{-1} = \frac{1}{1} = 1,\\ &4^{-\frac{1}{2}} = \frac{1}{4^{\frac{1}{2}}} = \frac{1}{\sqrt{4}} = \frac{1}{2},\\ &6^0 = 1,\\ &8^{\frac{2}{3}} = (8^{\frac{1}{3}})^2 = (\sqrt[3]{8})^2 = 2^2 = 4,\end{aligned}$$

and

$$16^{\frac{3}{4}} = (16^{\frac{1}{4}})^3 = (\sqrt[4]{16})^3 = 2^3 = 8.$$

We therefore see that only the expression of option B gives a number which is not an integer when it is evaluated.

For investigation

8.1 The standard convention is that for $x \neq 0$, we take 1 as the value of x^0. Why is this convention a sensible one to use?

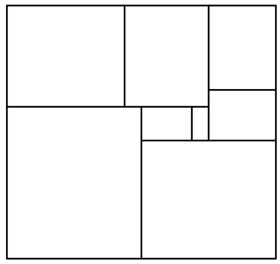
8.2 The standard convention is that 0^0 represents the number 1. Why is this a sensible convention?

8.3 In the answer to Question 8 we have used the fact that when p and q are positive integers, and $x > 0$, the convention is that $x^{\frac{p}{q}}$ means $(\sqrt[q]{x})^p$. Why do we adopt this convention?

9. The diagram shows an $n \times (n+1)$ rectangle tiled with $k \times (k+1)$ rectangles, where n and k are integers and k takes each value from 1 to 8 inclusive.

What is the value of n?

A 16 B 15 C 14 D 13 E 12



Solution **B**

The total area of the rectangles of size $k \times (k+1)$, for $k = 1, 2, 3, 4, 5, 6, 7, 8$, is

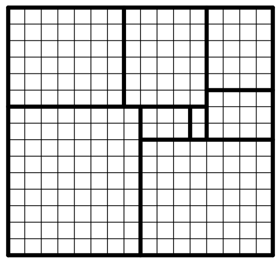
$$\begin{aligned} 1 \times 2 + 2 \times 3 + 3 \times 4 + 4 \times 5 + 5 \times 6 + 6 \times 7 + 7 \times 8 + 8 \times 9 &= 2 + 6 + 12 + 20 + 30 + 42 + 56 + 72 \\ &= 240 \\ &= 15 \times 16. \end{aligned}$$

Therefore $n = 15$.

In the context of the SMC the above calculation is sufficient to show that, if the smaller rectangles tile a rectangle of size $n \times (n+1)$, for some integer n, then $n = 15$.

However, for a complete solution it is necessary to show that the eight smaller rectangles can be used to a tile a 15×16 rectangle.

It looks from the figure in the question that this is possible. The figure on the right confirms that the sizes of the rectangles are correct.



Note also that from this figure we can see directly that the large rectangle has size 15×16.

For investigation

9.1 (a) Find a formula in terms of s for the total area of the rectangles of size $k \times (k+1)$ for all the integer values of k from 1 to s inclusive.

In other words, find a formula for the sum

$$1 \times 2 + 2 \times 3 + 3 \times 4 + \cdots + s \times (s+1).$$

[Note that using the Σ notation, we may write this sum as

$$\sum_{k=1}^{s} k \times (k+1), \text{ or, suppressing the multiplication sign, as } \sum_{k=1}^{s} k(k+1)].$$

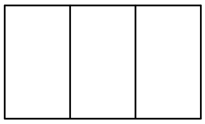
(b) Check that your formula gives the answer 240 when $s = 8$.

9.2 (a) Can you find values of s, other than $s = 8$, such that for some integer n

$$\sum_{k=1}^{s} k(k+1) = n(n+1)?$$

(b) For the values of s that you have found in answer to part (a) is it possible to use the rectangles of size $k \times (k+1)$, where k takes all integer values from 1 to s inclusive, to tile a rectangle of size $n \times (n+1)$?

10. A rectangle is divided into three smaller congruent rectangles as shown.



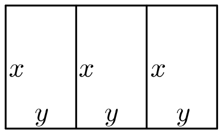
Each smaller rectangle is similar to the large rectangle.

In each of the four rectangles, what is the ratio of the length of a longer side to that of a shorter side?

A $2\sqrt{3} : 1$ B $3 : 1$ C $2 : 1$ D $\sqrt{3} : 1$ E $\sqrt{2} : 1$

Solution **D**

We suppose that the length of the longer sides of the three smaller rectangles is x and the length of their shorter sides is y.



It follows that the longer sides of the large rectangle have length $3y$, and that its shorter sides have length x.

Because the smaller rectangles are similar to the larger rectangle $\dfrac{x}{y} = \dfrac{3y}{x}$. Therefore $\dfrac{x^2}{y^2} = \dfrac{3}{1}$.
Hence $\dfrac{x}{y} = \dfrac{\sqrt{3}}{1}$.

It follows that the ratio of the length of a longer side to that of a shorter side in all the rectangles is $\sqrt{3} : 1$.

For investigation

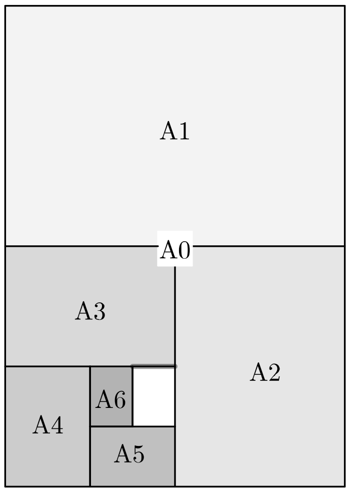
10.1 The A series of paper sizes, (A0, A1, A2, A3, . . .), is defined as follows.

The largest size is A0.

Two A1 sized sheets of paper are obtained by cutting an A0 sheet in half along a line parallel to its shorter edges.

Two A2 sized sheets of paper are obtained by cutting an A1 sheet in half in a similar way, and so on, as shown in the figure.

The shapes of all these sheets of paper are similar rectangles.



(a) What is the ratio of the length of a longer side to the length of the shorter side in all these rectangles?

(b) An A0 sheet of paper has area $1\,\text{m}^2$. What are lengths, to the nearest cm, of the longer and shorter sides of an A0 sheet of paper?

(c) The most commonly used of these sizes is A4. What are the lengths, to the nearest cm, of the longer and shorter sides of an A4 sheet of paper?

(d) Standard quality paper weighs $80\,\text{g}/\text{m}^2$. What is the weight of one standard quality sheet of A4 paper?

11. The teenagers Sam and Jo notice the following facts about their ages:

The difference between the squares of their ages is four times the sum of their ages.

The sum of their ages is eight times the difference between their ages.

What is the age of the older of the two?

A 15 B 16 C 17 D 18 E 19

Solution **D**

Suppose that the ages of the teenagers are a and b, with $a > b$.

Because the difference between the squares of their ages is four times the sum of their ages

$$a^2 - b^2 = 4(a + b).$$

By factorizing its left hand side, we may rewrite this last equation as

$$(a - b)(a + b) = 4(a + b)$$

Because $a + b \neq 0$, we may divide both sides of this last equation by $a + b$ to give

$$a - b = 4. \qquad (1)$$

Because the sum of their ages is eight times their difference

$$a + b = 8(a - b)$$

Hence, by (1)

$$a + b = 32. \qquad (2)$$

By adding equations (1) and (2), we obtain

$$2a = 36$$

and hence

$$a = 18.$$

Therefore the age of the older of the two teenagers is 18.

For investigation

11.1 What is the age of the younger teenager in Question 11?

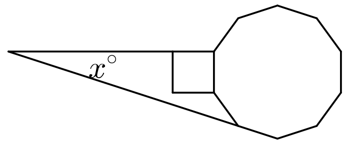
11.2 Suppose that the difference between the squares of the ages of two teenagers is six times the sum of their ages, and the sum of their ages is five times the difference of their ages.

What are their ages in this case?

11.3 Suppose that the difference between the squares of the ages of the two teenagers is k times the sum of their ages, and the sum of their ages in n times the difference of their ages.

Find a formula in terms of k and n for the ages of the teenagers. Check that your formula gives the correct answer to Question 11 and Problems 11.1 and 11.2.

12. The diagram shows a square and a regular decagon that share an edge. One side of the square is extended to meet an extended edge of the decagon.

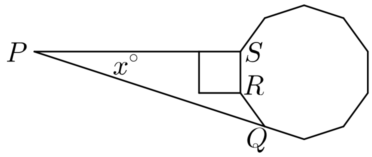


What is the value of x?

A 15 B 18 C 21 D 24 E 27

Solution **B**

We consider the quadrilateral $PQRS$ as shown in the figure on the right. Because $PQRS$ is a quadrilateral its angles have total 360°.



Because it is the exterior angle of a decagon, $\angle RQP = \frac{1}{10} \times 360^\circ = 36^\circ$.

It follows that the interior angle of a decagon is $(180 - 36)^\circ = 144^\circ$. Hence the reflex angle SRQ is $(360 - 144)^\circ = 216^\circ$. The angle PSR is an angle of the square and hence is 90°.

Therefore, we have

$$x^\circ + 36^\circ + 216^\circ + 90^\circ = 360^\circ.$$

It follows that

$$x^\circ = (360 - 342)^\circ = 18^\circ.$$

For investigation

12.1 We have used here the fact that the exterior angle of a regular polygon with n sides is $\frac{1}{n} \times 360^\circ$. Explain why this is true.

12.2 Prove that the sum of the angles of a quadrilateral is 360°.

13. Isobel: "Josh is innocent" Genotan: "Tegan is guilty"

Josh: "Genotan is guilty" Tegan: "Isobel is innocent"

Only the guilty person is lying; all the others are telling the truth.

Who is guilty?

A Isobel B Josh C Genotan D Tegan
E More information required

Solution **C**

There is only one guilty person, so either Genotan or Josh is lying. If Josh is lying, Genotan is innocent and is therefore telling the truth. Hence Tegan is guilty, contradicting the fact that there is just one guilty person. So Josh is not lying. Therefore Genotan is the guilty person.

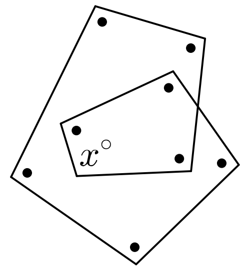
For investigation

13.1 Check that the guilt of Genotan is consistent with all the information given in the question.

14. In the diagram all the angles marked • are equal in size to the angle marked $x°$.

What is the value of x?

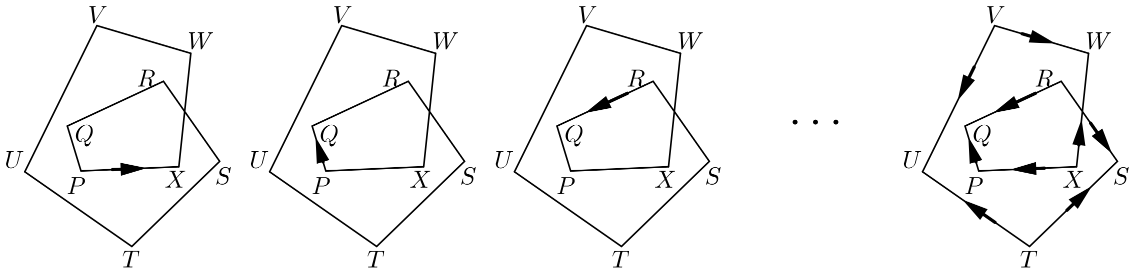
A 100 B 105 C 110 D 115 E 120



Solution **A**

Method 1

We label the vertices of the figure as shown below.



We place an arrow lying along PX in the direction shown in the first figure on the left above. We rotate this arrow anticlockwise about the point P until it lies along PQ in the direction shown in the second figure above. The arrow has been turned anticlockwise through the angle $x°$.

Next we rotate the arrow anticlockwise about the point Q until it lies along RQ in the direction shown in the third figure above. As all the angles marked • are $x°$, the arrow has again been turned anticlockwise through $x°$.

We continue this process, rotating the arrow anticlockwise through $x°$ about the points R, S, T, U, V, W and X in turn. The arrow ends up lying along XP in the direction shown in the figure on the right above.

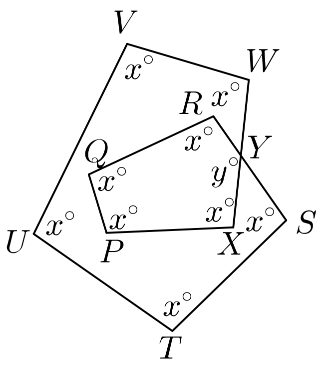
In this figure we have also shown the direction in which the arrow points on all the other edges during this process, apart from its initial position.

It will be seen that in this process the arrow has been turned through $2\frac{1}{2}$ complete revolutions. Therefore the total angle it has turned through is $2\frac{1}{2} \times 360°$, that is, through $900°$. In the process the arrow been rotated 9 times through the angle $x°$.

Therefore $9x = 900$ and hence $x = 100$.

Method 2

We let Y be the point where RS meets WX. We also let $\angle RYX = y°$.



The sum of the angles of a pentagon is 540°. In the pentagon $PXYRQ$, the interior angle at Y is $y°$ and all the other four angles are $x°$. Therefore we have

$$4x + y = 540. \quad (1).$$

The sum of the angles of a hexagon is 720°. In the hexagon $SYWVUT$, the reflex angle at Y is $360° - y°$ and all the other five angles are $x°$.

Therefore $5x + (360 - y) = 720$ and hence

$$5x - y = 360. \quad (2)$$

By adding equations (1) and (2) we deduce that $9x = 900$. Therefore $x = 100$.

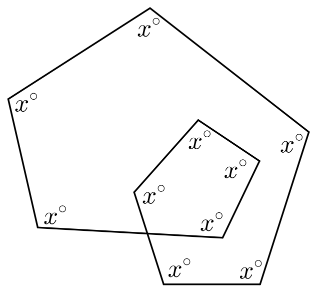
For investigation

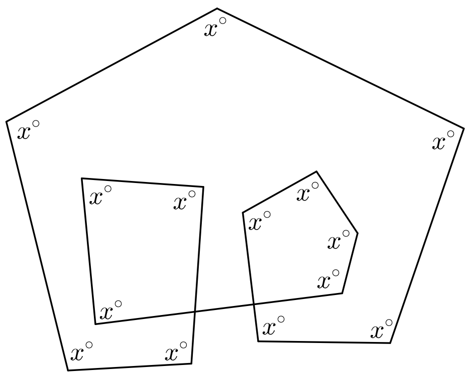
14.1 In Method 2 we have used the fact that the sum of the angles of a polygon with n edges is $(n - 2) \times 180°$, in the case $n = 5$ for the pentagon and $n = 6$ for the hexagon.

Prove that this formula is correct.

14.2 In Method 2 why is the reflex angle at Y of the hexagon $SYWVUT$ equal to $360° - y°$?

14.3





(a) In the figure on the left above there are 10 marked angles, all equal to $x°$. What is the value of x?

(b) In the figure on the right above there are 14 marked angles, all equal to $x°$. What is the value of x?

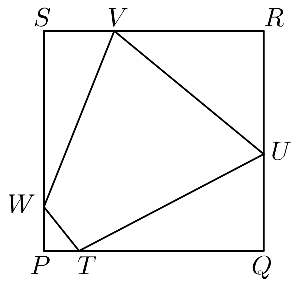
14.4 Generalize the results of Question 14 and Problem 14.3.

15. The diagram shows a square $PQRS$. Points T, U, V and W lie on the edges of the square, as shown, such that $PT = 1$, $QU = 2$, $RV = 3$ and $SW = 4$.

The area of $TUVW$ is half that of $PQRS$.

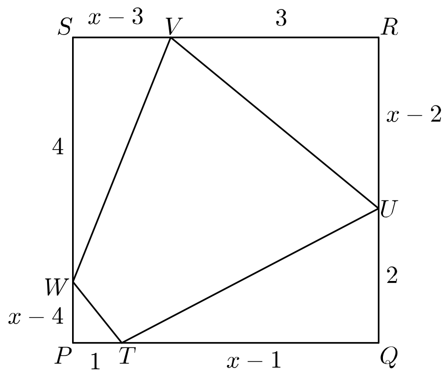
What is the length of PQ?

A 5 B 6 C 7 D 8 E 9



Solution **B**

We let the side length of the square $PQRS$ be x. Then the lengths of TQ, UR, VS and WP are $x-1$, $x-2$, $x-3$ and $x-4$, respectively.



The area of the square $PQRS$ is x^2. The area of $TUVW$ is a half of this. It follows that the sum of the areas of the triangles PTW, TQU, URV and VSW is also a half of the area of the square, and hence this sum equals $\frac{1}{2}x^2$.

The area of a triangle is half the product of its base and its height.

We therefore have

$$\tfrac{1}{2}(1 \times (x-4)) + \tfrac{1}{2}(2 \times (x-1)) + \tfrac{1}{2}(3 \times (x-2)) + \tfrac{1}{2}(4 \times (x-3)) = \tfrac{1}{2}x^2.$$

This equation simplifies to give

$$10x - 24 = x^2,$$

and therefore

$$x^2 - 10x + 24 = 0.$$

The left-hand side of the last equation factorizes to give

$$(x-4)(x-6) = 0.$$

Hence $x = 6$ or $x = 4$.

For there to be four triangles, as shown in the diagram, $x > 4$. We therefore deduce that $x = 6$.

So the length of PQ is 6.

For investigation

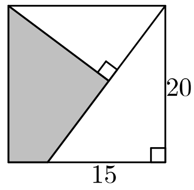
15.1 Suppose that, as in the question $PT = 1$, $QU = 2$, $RV = 3$ and $SW = 4$, but the area of $TUVW$ is two-thirds of the area of $PQRS$.

What is the length of PQ in this case?

16. The diagram shows two right-angled triangles inside a square. The perpendicular edges of the larger triangle have lengths 15 and 20.

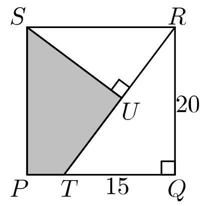
What is the area of the shaded quadrilateral?

A 142 B 146 C 150 D 154 E 158



Solution **D**

We let the vertices of the square be P, Q, R and S, and the points T and U be as shown.



We note first that, by Pythagoras' Theorem, applied to the right-angled triangle TQR,

$$RT^2 = 15^2 + 20^2 = 225 + 400 = 625 = 25^2.$$

It follows that $RT = 25$.

Because SR is parallel to PQ, the alternate angles, $\angle SRU$ and $\angle QTU$ are equal. Therefore, the right-angled triangles SUR and RQT are similar. Therefore their corresponding sides are in proportion. Hence

$$\frac{SR}{RT} = \frac{SU}{RQ} = \frac{RU}{TQ}.$$

Now $SR = RQ = 20$. Hence

$$\frac{20}{25} = \frac{SU}{20} = \frac{RU}{15}.$$

It follows that $SU = 16$ and $RU = 12$.

The area of the shaded quadrilateral is the area of the square $PQRS$ less the areas of the triangles SUR and RQT. It follows that the area of the shaded quadrilateral is

$$20^2 - \tfrac{1}{2}(16 \times 12) - \tfrac{1}{2}(20 \times 15) = 400 - 96 - 150 = 154.$$

Note

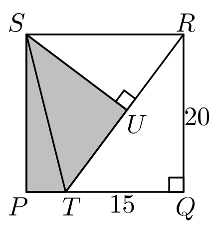
Another way to find the length of RT is to note that the lengths of TQ and QR are in the ratio $3 : 4$, and that therefore the right-angled triangle TQR is a $(3, 4, 5)$ triangle scaled by the factor 5. Hence the length of RT is $5 \times 5 = 25$.

Then, because the triangle RUS is similar to triangle TQR and has a hypotenuse of length 20, it follows that RUS is a $(3, 4, 5)$ triangle scaled by the factor 4. Hence RU has length $4 \times 3 = 12$ and SU has length $4 \times 4 = 16$.

For investigation

16.1 Calculate the area of the triangle SPT and the area of the triangle TUS.

Check that the sum of these areas is 154.



17. Amy, Beth and Claire each has some sweets. Amy gives one third of her sweets to Beth. Beth gives one third of all the sweets she now has to Claire. Then Claire gives one third of all the sweets she now has to Amy. All the girls end up having the same number of sweets.

Claire begins with 40 sweets.

How many sweets does Beth have originally?

A 20 B 30 C 40 D 50 E 60

SOLUTION **D**

We work backwards from the final situation when Amy, Beth and Claire end up with the same number of sweets. We let s be the number of sweets they all end up with.

Claire ends up with s sweets. Therefore, before she gave one third of her sweets to Amy, Claire had t sweets, where $\frac{2}{3}t = s$. This gives $t = \frac{3}{2}s$. We deduce that, before giving one-third of her sweets to Amy, Claire has $\frac{3}{2}s$ sweets and gives one third of these, namely $\frac{1}{2}s$ sweets, to Amy. Hence, before she received these sweets, Amy had $\frac{1}{2}s$ sweets.

Similarly, Beth ends up with s sweets after she has given one third of her sweets to Claire. Hence, before this she had $\frac{3}{2}s$ sweets. As Claire has $\frac{3}{2}s$ sweets after receiving $\frac{1}{2}s$ from Beth, she had s sweets before this.

It follows that after Amy has given one third of her sweets to Beth, Amy has $\frac{1}{2}s$ sweets, Beth has $\frac{3}{2}s$ sweets, and Claire has s sweets.

Therefore, before Amy gives one third of her sweets to Beth, Amy has $\frac{3}{4}s$ sweets. She gives $\frac{1}{4}s$ sweets to Beth. Hence, before receiving these, Beth had $\frac{3}{2}s - \frac{1}{4}s = \frac{5}{4}s$ sweets.

We can sum this up by the following table.

Stage	Amy	Beth	Claire
Final distribution of sweets, after Claire gives one third of her sweets ($\frac{1}{2}s$ sweets) to Amy.	s	s	s
Distribution of sweets after Beth gives one third of her sweets ($\frac{1}{2}s$ sweets) to Claire.	$\frac{1}{2}s$	s	$\frac{3}{2}s$
Distribution of sweets after Amy gives one third her sweets ($\frac{1}{4}s$ sweets) to Beth.	$\frac{1}{2}s$	$\frac{3}{2}s$	s
Initial distribution of sweets.	$\frac{3}{4}s$	$\frac{5}{4}s$	s

We are told that Claire begins with 40 sweets. Therefore $s = 40$. So Beth begins with $\frac{5}{4}(40) = 50$ sweets.

FOR INVESTIGATION

17.1 An alternative method is to suppose that, say, Amy begins with a sweets and Beth with b sweets, and then to work out how many sweets they all end up with. Use this method to answer Question 17. [Actually, to avoid fractions, it is better to assume Amy and Beth begin with $27a$ and $9b$ sweets, respectively.]

18. The arithmetic mean, A, of any two positive numbers x and y is defined to be $A = \frac{1}{2}(x+y)$ and their geometric mean, G, is defined to be $G = \sqrt{xy}$.

For two particular values x and y, with $x > y$, the ratio $A : G = 5 : 4$.

For these values of x and y, what is the ratio $x : y$?

A $5:4$ B $2:1$ C $5:2$ D $7:2$ E $4:1$

SOLUTION **E**

We are told that

$$\frac{\frac{1}{2}(x+y)}{\sqrt{xy}} = \frac{5}{4}.$$

Therefore

$$2(x+y) = 5\sqrt{xy}.$$

By squaring both sides of this equation we deduce that

$$4(x+y)^2 = 25xy.$$

By expanding the left-hand side of this last equation, we have

$$4(x^2 + 2xy + y^2) = 25xy,$$

or, equivalently,

$$4x^2 + 8xy + 4y^2 = 25xy.$$

Hence

$$4x^2 - 17xy + 4y^2 = 0.$$

The left-hand of this last equation factorizes to give

$$(4x - y)(x - 4y) = 0.$$

Hence

$$4x = y \text{ or } x = 4y.$$

It follows that, because $x > y$,

$$x = 4y.$$

Hence

$$x : y = 4 : 1.$$

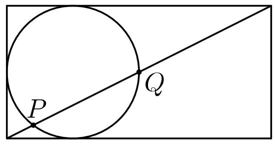
FOR INVESTIGATION

18.1 Suppose that x and y are positive integers, with $x > y$ and $A : G = 5 : 3$.

What is the ratio $x : y$ in this case?

18.2 Do there exist positive integers x and y such that $A < G$, that is, such that $\frac{1}{2}(x+y) < \sqrt{xy}$?

19. The diagram shows a circle of radius 1 touching three sides of a 2×4 rectangle. A diagonal of the rectangle intersects the circle at P and Q, as shown.

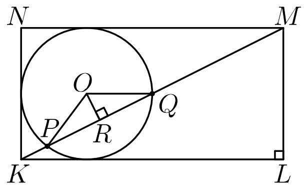


What is the length of the chord PQ?

A $\sqrt{5}$ B $\dfrac{4}{\sqrt{5}}$ C $\sqrt{5} - \dfrac{2}{\sqrt{5}}$ D $\dfrac{5\sqrt{5}}{6}$ E 2

Solution **B**

We let the the vertices of the rectangle be K, L, M, and N as shown in the figure. We let O be the centre of the circle, and R be the point where the perpendicular from O to the chord PQ meets the chord.



We note first that, by Pythagoras' Theorem applied to the right-angled triangle KLM, $KM^2 = KL^2 + LM^2 = 4^2 + 2^2 = 20$.

Therefore $KM = \sqrt{20} = 2\sqrt{5}$.

The radius OQ is parallel to KL. (You are asked to show this in Problem 19.1.) Therefore the alternate angles $\angle OQR$ and$\angle MKL$ are equal. It follows that the right-angled triangles OQR and MKL are similar. In particular,

$$\frac{RQ}{OQ} = \frac{KL}{KM}.$$

It follows that

$$RQ = \frac{KL}{KM} \times OQ = \frac{4}{2\sqrt{5}} \times 1 = \frac{2}{\sqrt{5}}.$$

The right-angled triangles ORQ and ORP are congruent as they share the side OR, and their hypotenuses OQ and OP are equal because they are radii of the same circle. It follows that $PR = RQ = \dfrac{2}{\sqrt{5}}$. We therefore conclude that

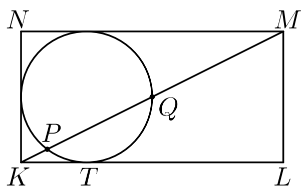
$$PQ = PR + RQ = \frac{2}{\sqrt{5}} + \frac{2}{\sqrt{5}} = \frac{4}{\sqrt{5}}.$$

For investigation

19.1 Explain why OQ is parallel to KL.

19.2 Let T be the point where the circle touches the edge KL.

Question 19 may also be solved by making use of the theorem (see Problem 19.3) which tells us that $KT^2 = KP \times KQ$.

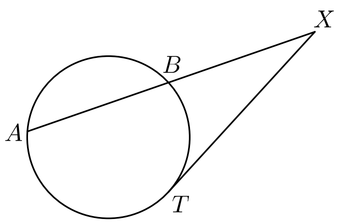


Show how this equation may be used to find the length of the chord PQ.

19.3 The theorem referred to in Problem 19.2 is the theorem which says that

If a chord AB of a circle and the tangent at the point T meet at a point X outside the circle, then

$$AX \times BX = TX^2.$$



Find a proof of this theorem.

You could find a proof for yourself, ask your teacher or look for a proof in a book or on the web.

You will find a discussion of this theorem on page 97 of the geometry book *Crossing the Bridge* by Gerry Leversha, published by UKMT.

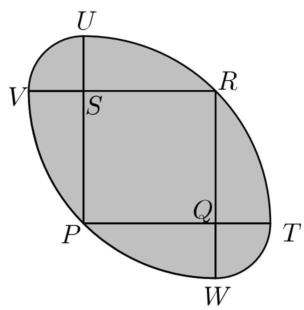
(Go to: http://shop.ukmt.org.uk/ukmt-books/)

20. The diagram shows a square $PQRS$ with edges of length 1, and four arcs, each of which is a quarter of a circle. Arc TRU has centre P; arc VPW has centre R; arc UV has centre S; and arc WT has centre Q.

What is the length of the perimeter of the shaded region?

A 6 B $(2\sqrt{2}-1)\pi$ C $(\sqrt{2}-\frac{1}{2})\pi$

D 2π E $(3\sqrt{2}-2)\pi$



Solution **B**

The square $PQRS$ has side length 1, and hence its diagonal PR has length $\sqrt{2}$. So the arc TRU with centre P has radius $\sqrt{2}$. The arc is a quarter of a circle. Hence its length is $\frac{1}{4} \times 2\pi\sqrt{2}$, that is $\frac{1}{2}\sqrt{2}\pi$.

Similarly, the length of the arc VPW is $\frac{1}{2}\sqrt{2}\pi$.

The radius of the arc UV is equal to the length of SU. Because PU has length $\sqrt{2}$ and PS has length 1, the length of SU is $\sqrt{2}-1$. The arc UV is one quarter of a circle with this radius. Hence the length of this arc is $\frac{1}{4} \times 2\pi(\sqrt{2}-1)$, that is $\frac{1}{2}(\sqrt{2}-1)\pi$.

Similarly, the arc WT has length $\frac{1}{2}(\sqrt{2}-1)\pi$.

Therefore the total length of the perimeter of the shaded region is given by

$$\tfrac{1}{2}\sqrt{2}\pi + \tfrac{1}{2}\sqrt{2}\pi + \tfrac{1}{2}(\sqrt{2}-1)\pi + \tfrac{1}{2}(\sqrt{2}-1)\pi = (2\sqrt{2}-1)\pi.$$

For investigation

20.1 What is the area of the shaded region?

21. How many pairs (x, y) of positive integers satisfy the equation $4^x = y^2 + 15$?

A 0 B 1 C 2 D 4
E an infinite number

Solution **C**

The equation $4^x = y^2 + 15$ may be rearranged as $4^x - y^2 = 15$. Now $4^x = (2^2)^x = (2^x)^2$. Hence $4^x - y^2$ may be factorized, using the standard factorization of the difference of two squares. This enables us to rewrite the equation as

$$(2^x - y)(2^x + y) = 15.$$

It follows that, for (x, y) to be a pair of positive integers that are solutions of the original equation, $2^x - y$ and $2^x + y$ must be positive integers whose product is 15, and with $2^x - y < 2^x + y$.

The only possibilities are therefore that either

$$2^x - y = 1 \quad \text{and} \quad 2^x + y = 15,$$

or

$$2^x - y = 3 \quad \text{and} \quad 2^x + y = 5.$$

In the first case $2^x = 8$ and $y = 7$, giving $x = 3$ and $y = 7$.

In the second case $2^x = 4$ and $y = 1$, giving $x = 2$ and $y = 1$.

Therefore there are just two pairs of positive integers that satisfy the equation $4^x = y^2 + 15$, namely, $(3, 7)$ and $(2, 1)$.

For investigation

21.1 (a) Check that if $2^x - y = 1$ and $2^x + y = 15$, then $2^x = 8$ and $y = 7$.

(b) Check that if $2^x - y = 3$ and $2^x + y = 5$, then $2^x = 4$ and $y = 1$.

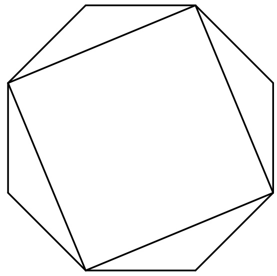
21.2 How many pairs of positive integers (x, y) are there which satisfy the equation $4^x = y^2 + 31$?

21.3 How many pairs of positive integers (x, y) are there which satisfy the equation $4^x = y^2 + 55$?

21.4 How many pairs of positive integers (x, y) are there which satisfy the equation $4^x = y^2 + 35$?

21.5 What can you say in general about those integers k for which there is at least one pair of positive integers (x, y) which satisfy the equation $4^x = y^2 + k$?

22. The diagram shows a regular octagon and a square formed by drawing four diagonals of the octagon. The edges of the square have length 1.

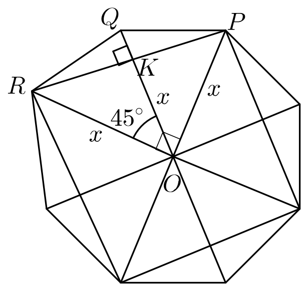


What is the area of the octagon?

A $\frac{\sqrt{6}}{2}$ B $\frac{4}{3}$ C $\frac{7}{5}$ D $\sqrt{2}$ E $\frac{3}{2}$

Solution **D**

Let O be the centre of the regular octagon, and let P, Q and R be adjacent vertices of the octagon as shown in the figure on the right. Let K be the point where OQ meets PR.



Let x be distance of O from the vertices of the octagon.

Since the edges of the square have length 1, $PR = 1$. By Pythagoras' Theorem applied to the right-angled triangle PRO, we have $x^2 + x^2 = 1^2$. Therefore $x^2 = \frac{1}{2}$ and hence $x = \frac{1}{\sqrt{2}}$.

The triangle ROQ has a base OQ of length x, that is $\frac{1}{\sqrt{2}}$, and height RK of length $\frac{1}{2}$. Therefore the area of the triangle ROQ is $\frac{1}{2} \times \frac{1}{\sqrt{2}} \times \frac{1}{2}$, which equals $\frac{1}{4\sqrt{2}}$.

The octagon is made up of 8 triangles each congruent to triangle ROQ.

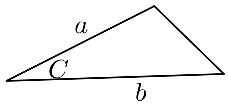
Therefore the area of the octagon is given by $8 \times \frac{1}{4\sqrt{2}} = \sqrt{2}$.

For investigation

22.1 The solution above assumes that $\angle POR = 90°$, $\angle RKO = 90°$ and $RK = \frac{1}{2}$. Explain why these statements are true.

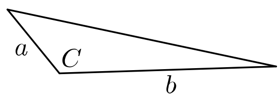
22.2 An alternative method for finding the area of the triangle ROQ is to use the "$\frac{1}{2}ab \sin C$" formula for the area of a triangle. Show how this also gives $\frac{1}{4\sqrt{2}}$ for the area of triangle ROQ.

22.3 (a) Show how the formula $\frac{1}{2}ab \sin C$ for the area of a triangle which has sides of lengths a and b, with included angle C, may be deduced from the fact that the area of a triangle is given by the formula



$$\text{area} = \tfrac{1}{2}(\text{base} \times \text{height}).$$

(b) Does your argument cover the case where C is an obtuse angle as well as the case where it is an acute angle?



23. The parabola with equation $y = x^2$ is reflected in the line with equation $y = x + 2$.

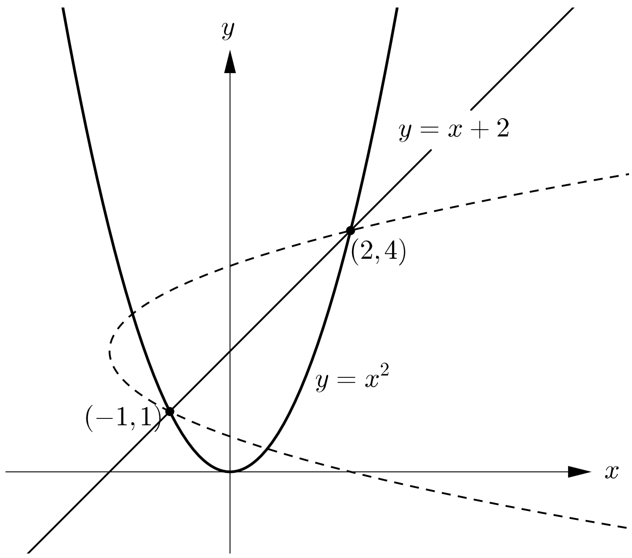
Which of the following is the equation of the reflected parabola?

A $x = y^2 + 4y + 2$ B $x = y^2 + 4y - 2$ C $x = y^2 - 4y + 2$
D $x = y^2 - 4y - 2$ E $x = y^2 + 2$

SOLUTION **C**

METHOD 1

The parabola with equation $y = x^2$ meets the line with equation $y = x + 2$ where $x^2 = x + 2$. This last equation may be rearranged as $x^2 - x - 2 = 0$ and then factorized as $(x + 1)(x - 2) = 0$. So its solutions are -1 and 2. For $x = -1$, we have $y = 1$ and for $x = 2$ we have $y = 4$. Therefore the parabola meets the line at the points with coordinates $(-1, 1)$ and $(2, 4)$ as shown in the figure on the right.



These two points remain where they are when they are reflected in the line. Therefore the reflected parabola also goes through these two points.

We can now test each equation given as an option to see if it is the equation of a curve which goes through the points $(-1, 1)$ and $(2, 4)$.

For example, because $-1 \neq 1^2 + 4 \times 1 + 2$ the coordinates of the point $(-1, 1)$ do not satisfy the equation $x = y^2 + 4y + 2$. It follows that $x = y^2 + 4y + 2$ is not the equation of a curve which goes through $(-1, 1)$. Therefore it is not the equation of the reflected parabola.

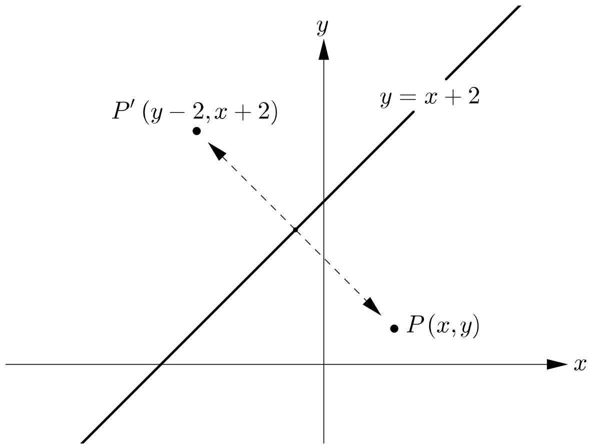
Because $-1 = 1^2 - 4 \times 1 + 2$ and $2 = 4^2 - 4 \times 4 + 2$, it follows that both the points $(-1, 1)$ and $(2, 4)$ lie on the curve with the equation $x = y^2 - 4y + 2$. It can be checked that these points do not lie on the curves given by any of the other equations.

Therefore, in the context of the SMC, we can conclude that the equation of the reflected parabola is $x = y^2 - 4y + 2$.

If we were not given the equations in the options we would need to calculate the equation of the reflected parabola. We adopt this approach in the second method.

METHOD 2

We leave it to the reader to check that when the point P with coordinates (x, y) is reflected in the line with equation $y = x + 2$, its image point P' has the coordinates $(y - 2, x + 2)$.



We put $x' = y - 2$ and $y' = x + 2$, so that $x = y' - 2$ and $y = x' + 2$

The point with coordinates (x', y') lies on the image under reflection of the parabola if, and only if, the point (x, y) lies on the parabola, that is, if, and only if, $y = x^2$. Therefore the condition that the point (x', y') lies on the image of the parabola is expressed by the equation $(x' + 2) = (y' - 2)^2$.

On expansion this equation may be written as $x' + 2 = y'^2 - 4y' + 4$. Rearranging this equation, and dropping the dashes, we deduce that the equation of the reflected parabola is

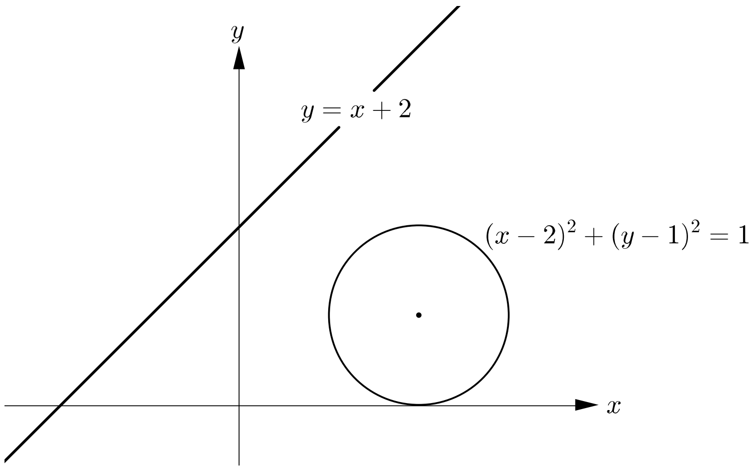
$$x = y^2 - 4y + 2.$$

FOR INVESTIGATION

23.1 Check that neither of the points $(-1, 1)$ and $(2, 4)$ lies on any of the curves with equations $x = y^2 + 4y + 2$, $x = y^2 + 4y - 2$, $x = y^2 - 4y - 2$, and $x = y^2 + 2$.

23.2 Show that when the point with coordinates (x, y) is reflected in the line with equation $y = x + 2$, its image is the point with coordinates $(y - 2, x + 2)$.

23.3



Find the equation of the circle that is obtained when the circle with centre $(2, 1)$ and radius 1 is reflected in the line with the equation $y = x + 2$.

[Note that the circle with centre $(2, 1)$ and radius 1 has the equation $(x-2)^2 + (y-1)^2 = 1$.]

23.4 Find the coordinates of the point that is obtained when the point with coordinates (x, y) is reflected in the line with equation $y = mx + c$.

24. There is a set of straight lines in the plane such that each line intersects exactly ten others.

Which of the following could not be the number of lines in that set?

A 11 B 12 C 15 D 16 E 20

SOLUTION **D**

It is convenient in this question to regard a line as being parallel to itself.

Suppose that there are l lines in the given set. Each line intersects all the lines in the set except those that are parallel to it. Therefore, as each line in the given set intersects 10 lines, each line in the set is parallel to $l - 10$ lines. We let $k = l - 10$.

Then $l = nk$, for some positive integer n. The nk lines in the set form n subsets each containing k parallel lines, and such that lines in different subsets are not parallel.

Each line intersects all but k of these nk lines. So each line intersects $nk - k$, that is, $(n - 1)k$ lines.

We are given that $(n - 1)k = 10$. Therefore $n - 1$ and k are positive integers whose product is 10. All the possible combinations of values of $n - 1$ and k with product 10 are shown in the table below. The third and fourth columns give the corresponding values of n and nk, the latter number being the total number of lines in the set.

$n-1$	k	n	nk
1	10	2	20
2	5	3	15
5	2	6	12
10	1	11	11

We see from this table that the only possibilities for the number of lines in a set of lines in which each line intersects 10 other lines are 11, 12, 15 and 20. In particular, 16 could not be the number of lines in the set.

FOR INVESTIGATION

24.1 Consider a set of lines in the plane such that each line intersects exactly 30 others. List the possibilities for the number of lines in this set.

24.2 Consider a set of lines in the plane such that each line intersects m others. What is the smallest possible number of lines in the set?

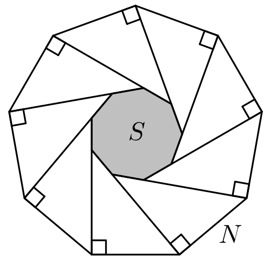
24.3 Consider a set of lines in the plane such that each line intersects m others. Show that the number of different possibilities for the number of lines in the set is equal to the number of different factors of m.

24.4 Consider a set of lines in the plane such that each line intersects 323 others. What is the largest possible number of lines in the set? [Note that $323 = 17 \times 19$].

24.5 Consider a set of lines in the plane such that each line intersects 360 others. What is the largest possible number of lines in the set?

24.6 Find a general method, in terms of m, for calculating the maximum possible number of lines in a set which is such that each line intersects m others.

25. The diagram shows a regular nonagon N. Moving clockwise around N, at each vertex a line segment is drawn perpendicular to the preceding edge. This produces a smaller nonagon S, shown shaded.



What fraction of the area of N is the area of S?

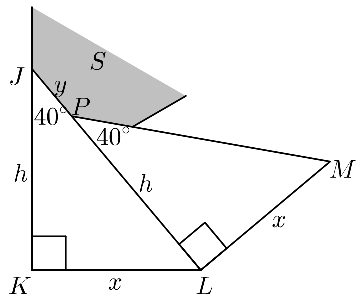
A $\dfrac{1-\cos 40°}{1+\cos 40°}$ B $\dfrac{\cos 40°}{1+\cos 40°}$ C $\dfrac{\sin 40°}{1+\sin 40°}$

D $\dfrac{1-\sin 40°}{1+\sin 40°}$ E $\dfrac{1}{9}$

Solution **A**

We let the side length of the larger regular nonagon N be x and the side length of the smaller regular nonagon S be y. Because S and N are similar figures, their areas are in the ratio $y^2 : x^2$. Therefore the required fraction, giving the area of S in terms of the area of N, is $\dfrac{y^2}{x^2}$.

The area between the nonagons is divided into nine triangles. Two of these adjacent triangles JKL and PLM are shown in the figure on the right.



The triangles JKL and PLM have right angles at K and L, respectively. The sides KL and LM each have length x as they are sides of the regular nonagon N. The angles at J and P in these triangles are exterior angles of S and therefore they are each $\frac{1}{9} \times 360°$, that is, 40°.

In the right-angled triangles JKL and PLM the angles are equal and $KL = LM$. Therefore the triangles are congruent. Hence $JK = PL$. We let h be the common length of JK and PL.

In the right-angled triangle JKL, the hypotenuse JL has length $y + h$. Therefore, applying Pythagoras' Theorem to this triangle, we have $x^2 + h^2 = (y + h)^2$. On expansion, this gives $x^2 + h^2 = y^2 + 2yh + h^2$, and hence $x^2 = y^2 + 2hy$. It follows that $\dfrac{x^2}{y^2} = \dfrac{y+2h}{y}$ and hence

$$\frac{y^2}{x^2} = \frac{y}{y+2h}. \quad (1)$$

From the triangle JKL we have $\dfrac{h}{y+h} = \cos 40°$. Hence $y + h = \dfrac{h}{\cos 40°}$ and therefore

$$y = \frac{h}{\cos 40°} - h \quad \text{and} \quad y + 2h = \frac{h}{\cos 40°} + h. \quad (2)$$

Substituting from (2) into the right-hand side of equation (1), we may now deduce that the area of S, as a fraction of the area of N, is

$$\frac{y^2}{x^2} = \frac{\dfrac{h}{\cos 40°} - h}{\dfrac{h}{\cos 40°} + h} = \frac{h - h\cos 40°}{h + h\cos 40°} = \frac{1-\cos 40°}{1+\cos 40°}.$$